1. Solve using method of variation of parameters

$$y'' + y = \frac{1}{1+\sin x}$$

Solution: Let the solution of the given DE be $y = y_c + y_p$ where y_c is the general solution for the homogenous portion $y'' + y = 0$ and y_p be a particular solution for the non-homogenous part.

Let $y = e^{mx}$ be a trial solution for $y'' + y = 0$. Then the auxillary equation is $m^2 + 1 = 0 \implies m = \pm i = 0 + i(\pm 1) ... (\alpha + i\beta)$. So using $y_c = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ we have

$$y_c = C_1 \cos x + C_2 \sin x \tag{1}$$

So from the equation (1) the first solution is $y_1 = y_1(x) = \cos x$ and the second solution is $y_2 = y_2(x) = \sin x$.

Now using the method of variation of parameters we have

$$y_p = v_1(x) y_1 + v_2(x) y_2 \tag{2}$$

where $v_1 = v_1(x), v_2 = v_2(x)$ are functions to be determined through

$$v_1' y_1 + v_2' y_2 = 0, \tag{3}$$

$$v_1' y_1' + v_2' y_2' = \frac{1}{1+\sin x}. \tag{4}$$

From (3), we have $v_1' = -\frac{y_2}{y_1} v_2'$. Substituting this into (4)

$$
\begin{aligned}
&\left(-\frac{y_2}{y_1}v_2'\right)y_1' + v_2'y_2' = \frac{1}{1+\sin x} \\
&\implies v_2'\left(y_2' - \frac{y_2}{y_1}y_1'\right) = \frac{1}{1+\sin x} \\
&\implies v_2'\left(\frac{y_1y_2' - y_2y_1'}{y_1}\right) = \frac{1}{1+\sin x} \\
&\implies v_2' = \frac{y_1}{(y_1y_2' - y_2y_1')(1+\sin x)} \\
&\implies v_1' = -\frac{y_2}{(y_1y_2' - y_2y_1')(1+\sin x)}
\end{aligned}
$$

Now

$$
y_1y_2' - y_2y_1' = \cos x(\cos x) - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1
$$

Hence,

$$
\begin{aligned}
v_2(x) &= \int \frac{\cos x}{1+\sin x}\,dx \qquad (5) \\
&= \int \frac{1}{1+u}\,du \quad (u = \sin x,\ du = \cos x\,dx) \\
&= \ln|1+\sin x| + C_3
\end{aligned}
$$

and

$$
\begin{aligned}
v_1(x) &= \int -\frac{\sin x}{1+\sin x}\,dx \qquad (6)\\
&= -\int \left(1 - \frac{1}{1+\sin x}\right) dx \\
&= -\int 1\,dx + \int \frac{1}{1+\sin x}\,dx \\
&= -x + \int \frac{1}{1+\sin x}\,dx \\
&= -x + \int \frac{\sec^2\left(\frac{x}{2}\right)}{\left(\tan\left(\frac{x}{2}\right)+1\right)^2}\,dx \\
&= -x + 2\int \frac{1}{u^2}\,du \quad \left(u = \tan\left(\frac{x}{2}\right)+1\right) \\
&= -x - \frac{2}{u} + C \\
&= -x - \frac{2}{\tan\left(\frac{x}{2}\right)+1} + C_4
\end{aligned}
$$

So

$$
y_p = \cos x\left(-x - \frac{2}{\tan\left(\frac{x}{2}\right)+1} + A\right) + \sin x\left(\ln|1+\sin x| + B\right)
$$

where A, B are particular constants. So the solution is

$$
y = y_c + y_p = C_1\cos x + C_2\sin x + \cos x\left(-x - \frac{2}{\tan\left(\frac{x}{2}\right)+1} + A\right) + \sin x\left(\ln|1+\sin x| + B\right).
$$

2. Solve using method of variation of parameters

$$
y'' + 3y' + 2y = \frac{1}{e^{2x}+1}
$$

Solution: Let the solution of the given DE be $y = y_c + y_p$ where y_c is the general solution for the homogenous portion $y'' + 3y' + 2y = 0$ and y_p be a particular solution for the non-homogenous part.

Let $y = e^{mx}$ be a trial solution for $y'' + 3y' + 2y = 0$. Then the auxillary equation is $m^2+3m+2 = 0 \implies (m+1)(m+2) = 0 \implies m = -1, -2$. So using $y_c = C_1e^{m_1x} + C_2e^{m_2x}$, we have

$$
y_c = C_1e^{-x} + C_2e^{-2x} \qquad (7)
$$

So from the equation (7) the first solution is $y_1 = y_1(x) = e^{-x}$ and the second solution is $y_2 = y_2(x) = e^{-2x}$.

Now using the method of variation of parameters we have

$$y_p = v_1(x)y_1 + v_2(x)y_2 \tag{8}$$

where $v_1 = v_1(x), v_2 = v_2(x)$ are functions to be determined through

$$v_1'y_1 + v_2'y_2 = 0, \tag{9}$$

$$v_1'y_1' + v_2'y_2' = \frac{1}{e^{2x}+1}. \tag{10}$$

From (9), we have $v_1' = -\frac{y_2}{y_1}v_2'$. Substituting this into (10)

$$\left(-\frac{y_2}{y_1}v_2'\right)y_1' + v_2'y_2' = \frac{1}{e^{2x}+1}$$

$$\implies v_2'\left(y_2' - \frac{y_2}{y_1}y_1'\right) = \frac{1}{e^{2x}+1}$$

$$\implies v_2'\left(\frac{y_1y_2' - y_2y_1'}{y_1}\right) = \frac{1}{e^{2x}+1}$$

$$\implies v_2' = \frac{y_1}{(y_1y_2' - y_2y_1')(e^{2x}+1)}$$

$$\implies v_1' = -\frac{y_2}{(y_1y_2' - y_2y_1')(e^{2x}+1)}$$

Now

$$y_1y_2' - y_2y_1' = e^{-x}(-2e^{-2x}) - e^{-2x}(-e^{-x}) = -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

Hence,

$$\begin{aligned} v_2(x) &= \int \frac{e^{-x}}{-e^{-3x}(e^{2x}+1)}\,dx \qquad (11) \\ &= \int -\frac{e^{2x}}{e^{2x}+1}\,dx \\ &= -\frac{1}{2}\int \frac{1}{u+1}\,du \quad (u = e^{2x},\ du = 2e^{2x}dx) \\ &= -\frac{1}{2}\ln|u+1| + C_3 \\ &= -\frac{1}{2}\ln(1+e^{2x}) + C_3 \end{aligned}$$

and

$$v_1(x) = \int \frac{e^{-2x}}{-e^{-3x}(e^{2x}+1)}\, dx \tag{12}$$
$$= \int \frac{e^{x}}{e^{2x}+1}\, dx$$
$$= \int \frac{1}{u^2+1}\, du \quad (u = e^x,\ du = e^x dx)$$
$$= \tan^{-1}(u) + C_4$$
$$= \tan^{-1}(e^x) + C_4$$

So
$$y_p = e^{-x}\left(\tan^{-1}(e^x) + A\right) + e^{-2x}\left(-\frac{1}{2}\ln(1+e^{2x}) + B\right)$$
where A, B are particular constants. So the solution is

$$y = C_1e^{-x} + C_2e^{-2x} + e^{-x}\left(\tan^{-1}(e^x) + A\right) + e^{-2x}\left(-\frac{1}{2}\ln(1+e^{2x}) + B\right).$$

3. Given that $y = e^{2x}$ is a solution of

$$(2x+1)y'' - (4x+1)y' + 4y = 0$$

find a second solution by reducing the order. Can a general solution be derived from these two solutions?

Solution: We first verify whether $y = e^{2x}$ satisfies the given differential equation.

Let $y = e^{2x} \implies y' = 2e^{2x}$ and $y'' = 4e^{2x}$. Substituting these into the given DE,

$$\begin{aligned}(2x+1)y'' - (4x+1)y' + 4y &= (2x+1)(4e^{2x}) - (4x+1)(2e^{2x}) + 4e^{2x} \\ &= e^{2x}\big[4(2x+1) - 2(4x+1) + 4\big] \\ &= e^{2x}\big[(8x+4) - (8x+2) + 4\big] \\ &= 6e^{2x} \neq 0\end{aligned}$$

So $y = e^{2x}$ is not a solution of the given DE and this cannot be solved by method of reduction of order.