Consider a mass m suspended from a spring attached to a rigid support. Assume gravity pulls the mass downward and the restoring force of the spring pulls the mass upward, that no other forces are in action and motion is considered in one dimension only. Suppose the mass is displaced from its equilibrium position and then it starts oscillating.

(a) Find the equation of displacement from equilibrium at time t.
(b) Find the amplitude of the motion.

Solution. Let $x = x(t)$ be the displacement from equilibrium and s be the static stretch of the spring at equilibrium. Then we have

$$m\frac{d^2x}{dt^2} = F_g + F_{\text{restoring}} = mg - k(s + x)$$

At equilibrium the mass is at rest, the displacement from equilibrium is zero, and the net force is zero. Thus $mg = ks$. Hence the differential equation becomes

$$m\frac{d^2x}{dt^2} = -kx$$

Let $\lambda^2 = \frac{k}{m}$. Then it can be written as

$$\frac{d^2x}{dt^2} + \lambda^2 x = 0$$

The auxiliary equation is $m^2 + \lambda^2 = 0 \implies m = \pm i\lambda$. So the general solution is

$$x(t) = C_1 \cos(\lambda t) + C_2 \sin(\lambda t)$$

where C_1, C_2 are arbitrary constants.

Now we wish to rewrite the solution as a function of only $\cos(\cdot)$.
Suppose $x(t) = C\cos(\lambda t - \phi) = C\cos(\lambda t)\cos\phi + C\sin(\lambda t)\sin\phi$.
Comparing coefficients, we have $C\cos\phi = C_1, C\sin\phi = C_2$. Hence the amplitude is $C = \sqrt{C_1^2 + C_2^2}$.
Suppose $x(0) = x_0$ and $x'(0) = v_0$. Substituting these in $x(t) = C\cos(\lambda t - \phi)$ we have, $x(0) = C\cos\phi$ and $x'(t) = -C\lambda\sin(\lambda t - \phi)$. So $v_0 = C\lambda\sin\phi$. Then C becomes $\sqrt{x_0^2 + \frac{v_0^2}{\lambda^2}}$.

Problem. A mass of 8 units is placed upon the lower end of a coil spring suspended from the ceiling. The mass comes to rest in its equilibrium position, thereby stretching the spring 6 units. The mass is then pulled down 3 units below its equilibrium position and released at $t = 0$ with an initial velocity of 1 unit, directed downward. Neglecting resistance and assuming no external forces, determine the equation for displacement and the amplitude of the motion.

Solution. Let $x(t)$ be the displacement from equilibrium and we take the downward direction to be positive. At equilibrium, we have $8 \cdot g = 6 \cdot k \implies k = \frac{4g}{3}$. The differential equation is therefore

$$8x'' + \frac{4g}{3}x = 0 \implies x'' + \frac{g}{6}x = 0$$

Let $\lambda^2 = \frac{g}{6}$. Then $x'' + \lambda^2 x = 0$.
Now the solution has the form $x(t) = C\cos(\lambda t - \phi)$.
So $x(0) = C\cos\phi = 3$ and $x'(t) = -C\lambda\sin(\lambda t - \phi) \implies v_0 = x'(0) = -C\lambda\sin(-\phi) = C\lambda\sin\phi = 1$.
So $C = \sqrt{9 + \frac{6}{g}}$ and $x(t) = C\cos\left(\sqrt{\frac{g}{6}}\,t - \phi\right)$ where $\tan\phi = \frac{1}{3\lambda}$.