First Order Linear Partial Differential Equations

Let $z = z(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ be an unknown function of two variables x and y. A first-order partial differential equation (PDE) is given by

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0, \tag{1}$$

where $F : U \times \mathbb{R}^3 \to \mathbb{R}$.

$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ is commonly denoted as p and q respectively.

A function $z = \phi(x, y) : I \subset \mathbb{R}^2 \to \mathbb{R}$ is called a solution of (1) if the following holds:

1. $(x, y, z, p, q) \in U \times \mathbb{R}^3$ for all $(x, y) \in I$.
2. $z \in C^1(I)$ i.e. it is continuously differentiable.
3. Putting the solution in the function yields an identity $F\left(x, y, \phi(x, y), \phi_x, \phi_y\right) = 0$.

A first-order linear PDE has the general form $a(x, y)p + b(x, y)q + c(x, y)z = d(x, y)$.

Formation of PDE

Form linear first order PDEs by eliminating arbitrary constants (or functions). Here $z = z(x, y)$.

(i) $z = ax + by + ab$

(ii) $z = ax^2 + by^2$

(iii) $z = (x + a)(y + b)$

(iv) $z = xy + y\sqrt{x^2 - a^2 + b^2}$

(v) $z = x^n f\left(\frac{y}{x}\right)$

Solution

(i) Differentiating the given equation partially with respect to x and y, we have,

$$p = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(ax + by + ab) = a, \qquad q = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(ax + by + ab) = b.$$

Substituting these into the given equation we have the required first order linear PDE $z = px + qy + pq$.

(ii) Differentiating the given equation partially with respect to x and y, we have,

$$p = \frac{\partial z}{\partial x} = 2ax, \qquad q = \frac{\partial z}{\partial y} = 2by.$$

Thus $a = \frac{p}{2x}$ and $b = \frac{q}{2y}$. Substituting into the given equation we get the required PDE $2z = px + qy$.

(iii) Differentiating the given equation partially with respect to x and y, we have,

$$p = \frac{\partial z}{\partial x} = y + b, \qquad q = \frac{\partial z}{\partial y} = x + a.$$

Thus $a = q - x$ and $b = p - y$. Substituting into the given equation we get the required PDE $z = pq$.

(iv) Differentiating the given equation partially with respect to x and y, we have,

$$p = \frac{\partial z}{\partial x} = y + y \cdot \frac{x}{\sqrt{x^2 - a^2 + b^2}}, \qquad q = \frac{\partial z}{\partial y} = x + \sqrt{x^2 - a^2 + b^2}.$$

Thus

$$p - y = \frac{xy}{\sqrt{x^2 - a^2 + b^2}} \implies \sqrt{x^2 - a^2 + b^2} = \frac{xy}{p - y}.$$

Substituting into the expression for q, we get

$$q - x = \frac{xy}{p - y}.$$

Eliminating, we obtain the required PDE $px + qy = pq$.

(v) Differentiating the given equation partially with respect to x and y, we have,

$$p = \frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right),$$

$$q = \frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right).$$

Thus,

$$px = -x^{n-1} y f'\left(\frac{y}{x}\right) + n x^n f\left(\frac{y}{x}\right), \qquad qy = x^{n-1} y f'\left(\frac{y}{x}\right).$$

Adding, we obtain $px + qy = nx^n f\left(\frac{y}{x}\right) = nz$, which is the required PDE.

Lagrange's first order Linear PDE

Let $z = z(x(t), y(t))$. Then, by the chain rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Now suppose $u = u(x, y, z)$, where $z = z(x, y)$. We introduce a parameter t and consider $x = x(t)$, $y = y(t)$, so that $z = z(x(t), y(t))$.
Then by the same chain rule as above,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Using

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt},$$

we have

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)\frac{dy}{dt}.$$

Again, as u depends only on x and y, we have, by the chain rule,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

Comparing the coefficients of $\frac{dx}{dt}$ and $\frac{dy}{dt}$, we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y}.$$

Now suppose $F(u, v) = 0$ is an unknown function in u, v where $u = u(x, y, z)$ and $v = v(x, y, z)$ and $z = z(x, y)$.

Then we have,

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.$$

Putting the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ from above and similarly for $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ we have

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right), \\
\frac{\partial F}{\partial y} &= \frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right).
\end{aligned}$$

Now the above is a simultaneous linear equation and we want to eliminate F_u, F_v. Thus,

$$\begin{vmatrix} u_x + u_z z_x & v_x + v_z z_x \\ u_y + u_z z_y & v_y + v_z z_y \end{vmatrix} = 0.$$

Expanding the determinant, we obtain

$$\begin{aligned}
&(u_x + u_z z_x)(v_y + v_z z_y) - (u_y + u_z z_y)(v_x + v_z z_x) = 0 \\
&\implies (u_x v_y - u_y v_x) + (u_x v_z - u_z v_x) z_y + (u_z v_y - u_y v_z) z_x = 0.
\end{aligned}$$

Thus, rewriting as,

$$Pp + Qq = R,$$

where

$$p = z_x, \quad q = z_y, \quad P = u_z v_y - u_y v_z, \quad Q = u_x v_z - u_z v_x, \quad R = u_y v_x - u_x v_y.$$

is called Lagrange's first order linear PDE.

The general solution of Lagrange's first order linear PDE can be written in the form $F(u, v) = 0$ where $u = u(x, y, z) = C_1, v = v(x, y, z) = C_2$, $z = z(x, y)$ and C_1, C_2 are arbitrary constants. This is the reason we formed the differential equation using such a function.

To solve Lagrange's first order linear PDE, we first find the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Now a few methods exist.

(i) Method of grouping

We take any two ratios such as

$$\frac{dx}{P} = \frac{dy}{Q}.$$

Solving this, we obtain a relation

$$u(x, y, z) = c_1.$$

Similarly, taking another pair, for example,

$$\frac{dy}{Q} = \frac{dz}{R},$$

we obtain another relation

$$v(x, y, z) = c_2.$$

Thus, the general solution is given by

$$F(u, v) = 0$$

(ii) Method of multipliers

We choose functions (called multipliers) $l = l(x), m = m(y), n = n(z)$ such that

$$lP + mQ + nR = 0.$$

Then,

$$l\,dx + m\,dy + n\,dz = 0.$$

Integrating, we obtain a relation

$$u(x, y, z) = c_1.$$

Similarly, by choosing another set of multipliers $l = l_1(x), m = m_1(y), n = n_1(z)$ satisfying

$$l_1 P + m_1 Q + n_1 R = 0,$$

we obtain another relation

$$v(x, y, z) = c_2.$$

Hence, the general solution is

$$F(u, v) = 0$$

(ii) Method of ratio manipulation

We manipulate the ratios to form simpler differential relations.
For example, if

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

then we may write

$$\frac{dx - dy}{P - Q} = \frac{dy - dz}{Q - R} = \frac{dz - dx}{R - P}.$$

(addition can be used similarly). Solving any two of these, we obtain two independent relations

$$u(x, y, z) = c_1, \qquad v(x, y, z) = c_2.$$

Hence, the general solution is

$$F(u, v) = 0.$$

Problem. Find the general solution of the given PDEs below. Here $z = z(x, y),\ p = z_x,\ q = z_y$.

(i) $px + qy = z$

(ii) $p \sin x + q \cos y = \tan z$

(iii) $y^2 p - xyq = x(z - 2y)$

(iv) $pz - qz = z^2 + (x + y)^2$

(v) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

(vi) $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

(vii) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

Solution

(i) The auxiliary equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

From $\frac{dx}{x} = \frac{dy}{y}$, we get

$$\log x - \log y = c_1 \implies u = \left|\frac{x}{y}\right| = c_1.$$

From $\frac{dx}{x} = \frac{dz}{z}$, we get

$$\log x - \log z = c_2 \implies v = \left|\frac{x}{z}\right| = c_2.$$

Hence, the general solution is $F(u, v) = 0$.

(ii) The auxiliary equation is

$$\frac{dx}{\sin x} = \frac{dy}{\cos y} = \frac{dz}{\tan z}.$$

From $\frac{dx}{\sin x} = \frac{dz}{\tan z}$, we have

$$\int \csc x\, dx = \int \cot z\, dz \implies \log\tan\frac{x}{2} = \log\sin z + \log c_1,$$

so $u = \left|\frac{\tan(x/2)}{\sin z}\right| = c_1$.
From $\frac{dy}{\cos y} = \frac{dz}{\tan z}$, we get

$$\int \sec y\, dy = \int \cot z\, dz \implies \log(\sec y + \tan y) = \log\sin z + \log c_2,$$

so $v = \left|\frac{\sec y + \tan y}{\sin z}\right| = c_2$.
Hence, the general solution is $F(u, v) = 0$.

(iii) The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}.$$

The first two ratios give

$$\frac{dx}{y^2} = \frac{dy}{-xy} \implies y\,dx = -x\,dy \implies u = x^2 + y^2 = C_1.$$

The last two ratios give

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \implies (z-2y)\,dy = -y\,dz \implies v = yz - y^2 = C_2.$$

Hence, the general solution is $F(u, v) = 0$.

(iv) The auxiliary equation is

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x+y)^2}.$$

The first two ratios give

$$\frac{dx}{z} = \frac{dy}{-z} \implies dx + dy = 0 \implies u = x + y = C_1.$$

Now, considering $x + y = C_1$ and the first and last ratios, we obtain

$$\frac{dx}{z} = \frac{dz}{z^2 + C_1^2} \implies \frac{2z\,dz}{z^2 + C_1^2} = 2\,dx.$$

Integrating,

$$v = \ln(z^2 + C_1^2) - 2x = C_2.$$

Hence, the general solution is $F(u, v) = 0$.

(v) The auxiliary equation is

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}.$$

which can be rewritten as

$$\frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}.$$

Considering the first two terms and integrating, we get

$$\frac{dx - dy}{x - y} = \frac{d(x - y)}{x - y} = d\big(\ln(x - y)\big),$$

and

$$\frac{dx - dy}{x + y + z} = d\big(\ln(x + y + z)\big),$$

which gives

$$\ln\left(\frac{x - y}{x + y + z}\right) = C_1 \quad \Longrightarrow \quad u = \left|\frac{x - y}{x + y + z}\right| = C_1.$$

Similarly, considering the last two terms and integrating, we obtain

$$v = \left|\frac{y - z}{z - x}\right| = C_2.$$

Hence, the general solution is $F(u, v) = 0$.

(vi) The auxiliary equation is

$$\frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)}.$$

Using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

Integrating,

$$\log x + \log y + \log z = \log c_1 \implies u = |xyz| = c_1.$$

Similarly, using $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ as multipliers, we have,

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0.$$

Integrating,

$$v = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_2.$$

Hence, the general solution is $F(u, v) = 0$.

(vii) The auxiliary equation is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}.$$

Using x, y, z as multipliers, we get

$$x\,dx + y\,dy + z\,dz = 0.$$

Integrating,

$$u = x^2 + y^2 + z^2 = c_1.$$

Again, using $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

Integrating,

$$\log x + \log y + \log z = \log c_2 \implies v = |xyz| = c_2.$$

Hence, the general solution is $F(u, v) = 0$.

Problem. Form the first order linear PDE from these equations

(i) $z = ax + by + a^2 + b^2$

(ii) $z = a + b(x + y)$

(iii) $ax + by + cz = 1$

(iv) $z = f\left(\frac{xy}{z}\right)$

Problem. Find the general solution of the given PDEs below. Here $z = z(x, y),\ p = z_x,\ q = z_y$.

(i) $p \tan x + q \tan y = \tan z$

(ii) $p - q = \log(xy)$

(iii) $(y - z)p + (x - y)q = (z - x)$

(iv) $(x + 2z)p + (4xz - y)q = 2x^2 + y$

(v) $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$