First order Linear differential equation

Example 1. Solve $(4x^2y - 6)dx + x^3dy = 0,\ (x \neq 0)$

Solution. The equation is first rewritten to be a linear differential equation in y.

$$
\begin{aligned}
&(4x^2y - 6)dx + x^3dy = 0, \quad (x \neq 0)\\
&\implies x^3\frac{dy}{dx} + 4x^2y - 6 = 0\\
&\implies x^3\frac{dy}{dx} + 4x^2y = 6\\
&\implies \frac{dy}{dx} + \frac{4}{x}y = \frac{6}{x^3} \qquad (1)
\end{aligned}
$$

Now equation (1) is of the form

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

So it has an integrating factor of the form $e^{\int P(x)dx}$. Putting $P(x)$ and evaluating the integral, $\text{IF} = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$.

Multiplying both sides of equation (1) by the integrating factor and integrating

$$
\begin{aligned}
&\implies x^4\cdot\frac{dy}{dx} + 4x^3\cdot y = 6x\\
&\implies d\big(x^4y\big) = 6x\,dx\\
&\implies x^4y = 3x^2 + C\\
&\implies y = \frac{3}{x^2} + \frac{C}{x^4} \quad \text{...which is the required solution.}
\end{aligned}
$$

Example 2. Solve $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0, (y \neq 0)$

Solution. The equation is first rewritten to be a linear differential equation in x.

$$
\begin{aligned}
& y^2+\left(x-\frac{1}{y}\right)\frac{dy}{dx}=0, \quad (y\neq 0)\\
& \implies y^2=\left(\frac{1}{y}-x\right)\frac{dy}{dx}\\
& \implies \frac{dx}{dy}=\frac{1}{y^3}-\frac{x}{y^2}\\
& \implies \frac{dx}{dy}+\frac{1}{y^2}\cdot x=\frac{1}{y^3} \qquad (2)
\end{aligned}
$$

Now equation (2) is of the form

$$\frac{dx}{dy}+P(y)\cdot x=Q(y)$$

So it has an integrating factor of the form $\mathrm{e}^{\int P(y)dy}$. Putting $P(y)$ and evaluating the integral, $\text{IF}=\mathrm{e}^{\int \frac{1}{y^2}dy}=\mathrm{e}^{-\frac{1}{y}}$.

Multiplying both sides of equation (2) by the integrating factor and integrating

$$
\begin{aligned}
& \implies \mathrm{e}^{-\frac{1}{y}}\cdot\frac{dx}{dy}+\mathrm{e}^{-\frac{1}{y}}\cdot\frac{1}{y^2}\cdot x=\mathrm{e}^{-\frac{1}{y}}\cdot\frac{1}{y^3}\\
& \implies d\Big(x\cdot\mathrm{e}^{-\frac{1}{y}}\Big)=\mathrm{e}^{-\frac{1}{y}}\cdot\frac{1}{y^3}\,dy\\
& \implies x=\frac{(y+1)}{y}+C\mathrm{e}^{\frac{1}{y}} \quad \text{...which is the required solution.}
\end{aligned}
$$

Reducible to First order Linear differential equation

(i) Bernoulli equation

Example 3. Solve $y(2xy+e^x)dx-e^xdy=0,(y\neq 0)$

Solution. The equation is first rewritten to be a Bernoulli differential equation in y.

$$
\begin{aligned}
& y(2xy+e^x)dx-e^xdy=0,(y\neq 0)\\
& \implies e^x\frac{dy}{dx}=y(2xy+e^x)\\
& \implies \frac{dy}{dx}=y+2xe^{-x}y^2\\
& \implies \frac{dy}{dx}-y=2xe^{-x}y^2 \qquad (3)
\end{aligned}
$$

Now equation (3) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where $P(x) = -1$, $Q(x) = 2xe^{-x}$ and $n = 2$.
Putting $v = y^{1-n} = y^{-1}$, so that

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}.$$

Multiplying equation (3) by $-y^{-2}$ and using the substitution,

$$\begin{aligned} & -y^{-2}\frac{dy}{dx} + y^{-1} = -2xe^{-x} \\ \implies & \frac{dv}{dx} + v = -2xe^{-x} \end{aligned} \tag{4}$$

Now equation (4) is a linear differential equation in v of the form

$$\frac{dv}{dx} + P(x)v = Q(x).$$

So it has an integrating factor of the form $\mathrm{e}^{\int P(x)dx}$. Putting $P(x)$ and evaluating the integral, $\mathrm{IF} = \mathrm{e}^{\int 1\,dx} = \mathrm{e}^x$.

Multiplying both sides of equation (4) by the integrating factor and integrating

$$\begin{aligned} \implies & \mathrm{e}^x\frac{dv}{dx} + \mathrm{e}^x v = -2x \\ \implies & d(v\mathrm{e}^x) = -2x\,dx \\ \implies & v\mathrm{e}^x = -x^2 + C \\ \implies & v = (C - x^2)\mathrm{e}^{-x} \\ \implies & \frac{1}{y} = (C - x^2)\mathrm{e}^{-x} \\ \implies & y = \frac{\mathrm{e}^x}{C - x^2} \quad \text{...which is the required solution.} \end{aligned}$$

Exercise:

Reduce the differential equations below to a first order linear differential equation and find its solution.

1. $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$ Hint. Bernoulli DE

2. $\frac{dx}{dt} + \frac{x(t+1)}{2t} = \frac{t+1}{xt}$ Hint. Bernoulli DE

3. $e^x \left[y - 3(e^x + 1)^2 \right] dx + (e^x + 1) dy = 0,\ y(0) = 4$ Hint. Divide by $(e^x + 1)$

4. $\sin y \frac{dy}{dx} = \cos x (2 \cos y - \sin^2 x)$ Hint. Put $z = \cos y$